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The Generalization of the Keldysh Theorem

MILUTIN R. DOSTANIĆ

Institute for Applied Mathematics and Electronics, Kneza Miloša 37, Belgrade, Yugoslavia

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In this paper we generalize the theorem by M. V. Keldysh on completeness of a system root vectors of a weakly perturbed m-sectorial compact operator with nontrivial kernel.

1. INTRODUCTION AND NOTATION

In this paper we generalize the theorem by M. V. Keldysh [4] on completeness of a system root vectors of weakly perturbed injective, compact selfadjoint operators H to the case of weakly perturbed m-sectorial compact operator with nontrivial kernel.

Let $\mathscr{H}$ be a separable Hilbert space over $\mathbb{C}$, and $A \in \mathscr{L}(\mathscr{H})$ be a compact operator. The eigenvalues of the compact nonnegative selfadjoint operator $(A^*A)^{1/2}$ arranged in decreasing order and repeated according to multiplicity, form a sequence $s_1, s_2, s_3, \ldots$ approaching zero. Following V. Neumann Schatten [1, Sect. XI, 9] we say that A belongs to the class C_p $(0 < p < \infty)$ provided $\sum_j s_j^p < \infty$. The set C_p is a two sided ideal in $\mathscr{L}(\mathscr{H})$. The number $\lambda \in \mathbb{C}$ is called a characteristic value of A if there exists $x \neq 0$ $(x \in \mathscr{H})$ such that $(I - \lambda A)x = 0$, of course $\lambda \neq 0$ and x is an eigenvector of A for the eigenvalue λ^{-1}. Further $x \neq 0$ is a generalized eigenvector (for the characteristic value λ) if for some $k \geqslant 1$

$$(I - \lambda A)^k x = 0.$$

Let $\overline{\mathscr{R}(A)}$ denote the closure of the range of A. In [3], the generalization of the Keldysh theorem to the case of weakly perturbed selfadjoint operators with nontrivial kernel under some additional assumption is given.

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2. Main Result

THEOREM. *Let* $B = A(I+S)$ *where* S *is a compact operator* $-1 \in \rho(S)$, *and* $\operatorname{Ker} A \subset \operatorname{Ker} S$. *If the operator* $A \in C_p$ $(0 < p < \infty)$ *is such that the values of quadratic form* (Af, f) *lie in the sector*

$$\Omega = \left\{\lambda : |\arg \lambda| \leqslant \theta < \frac{\pi}{2\rho'}\right\}, \qquad \text{where} \quad \rho' > \max\{1, p\},$$

then:

(a) *For all* ε, $0 < \varepsilon < \pi/2 - \theta$, *all characteristic values of operator* B, *except finitely many, lie in the sector* $F_\varepsilon = \{\lambda : |\arg \lambda| < \theta + \varepsilon\}$.

(b) *The system of generalized eigenvectors of the operator* B *is complete in* $\overline{\mathscr{R}(A)}$.

Before we prove this theorem first we show the following

LEMMA. *Under the assumptions given in the previous theorem*

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in F_\varepsilon^c}} \|S(I - \lambda A)^{-1}\| = 0, \tag{1}$$

where $F_\varepsilon^c = \mathbb{C} \backslash F_\varepsilon$ *and convergence is uniform on* F_ε^c.

Proof. Because of given assumptions $\{(Af, f) : f \in \mathscr{H}\} \subset \Omega$ so [2] holds $\|(A - \mu I)^{-1}\| \leqslant 1/d(\mu, W_A)$ $(\mu \notin W_A)$, where W_A is the closure of set $\{(Af, f) : \|f\| = 1\}$ and $d(\mu, W_A)$ is the distance between point μ and set W_A.

If $\lambda \in F_\varepsilon^c$ for $\mu = \lambda^{-1}$ we have $\mu \in F_\varepsilon^c$ and one has

$$\begin{aligned} \|(I - \lambda A)^{-1}\| &\leqslant \frac{1}{\sin \varepsilon} \\ \|(I - \bar{\lambda} A^*)^{-1}\| &\leqslant \frac{1}{\sin \varepsilon}. \end{aligned} \tag{2}$$

Since the operator S is compact it can be represented in the form

$$S = \sum_k \mu_k(\cdot, f_k)\, g_k \qquad (\mu_1 = \|s\| \geqslant \mu_2 \geqslant \mu_3 \geqslant \cdots).$$

The system of vectors $\{g_k\}$ and $\{f_k\}$ are orthonormal systems, so that

$$S^* g_k = \mu_k f_k. \tag{3}$$

The assumptions of the Theorem imply that $\overline{\mathscr{R}(A^*)} \supset \overline{\mathscr{R}(S^*)}$. This inclusion and (3) imply that $f_k \in \overline{\mathscr{R}(A^*)}$ for every $k \in N$. Since S can be uniformly approximated by finite rank operators, to prove the Lemma it is enough to show (1) for an operator S of the form

$$S = \sum_{k=1}^{n} \mu_k(\cdot, f_k)\, g_k.$$

But, in order to show that it is enough to consider S of the form

$$S = \mu_k(\cdot, f_k)\, g_k \qquad (\forall k \in N).$$

For $S = \mu_k(\cdot, f_k)\, g_k$ it is easy to see that

$$\|S(I - \lambda A)^{-1}\| \leqslant \|(I - \bar{\lambda} A^*)^{-1} f_k\| \cdot \|S\| \qquad (\text{for } \lambda \in F_\varepsilon^c).$$

So, it is enough to prove that

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in F_\varepsilon^c}} \|(I - \bar{\lambda} A^*)^{-1} f_k\| = 0 \qquad (\forall k \in N) \text{ uniformly on } F_\varepsilon^c. \tag{4}$$

Now, one can apply the theorem of Lidsky [5].

Let $h \in \overline{\mathscr{R}(A^*)}$. Thus for every fixed α (such that $\max\{1, p\} \leqslant \alpha < \pi/2\theta$) the Fourier series of the vector h with respect to the system of main vectors $\{G_s\}$ (in the sense of [5]) of A^* is summable by Abel's method of order α. So

$$h = \lim_{t \to 0+} \sum_{\nu=1}^{\infty} \left(\sum_{s=N_\nu+1}^{N_{\nu+1}} C_s(t)\, G_s \right). \tag{5}$$

(The above is the fundamental theorem in [5]. The coefficients $C_s(t)$ are defined in [5].) The series in (5) is convergent in $\mathscr{H}$ for every fixed $t > 0$.

From (5) it follows that the system of vectors is complete in $\overline{\mathscr{R}(A^*)}$. Since $f_k \in \overline{\mathscr{R}(A^*)}$, the lemma follows from (2), (4), (5) and

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in F_\varepsilon^c}} (I - \bar{\lambda} A^*)^{-1} G_s = 0 \qquad (\forall s \in N) \text{ uniformly on } F_\varepsilon^c. \tag{6}$$

The system of vectors $\{G_s\}$ is the union of Jordan's chains of main vectors of operator A^* [5] which correspond to different eigenvalues of A^*. So, it is enough to prove (6) for one such chain.

Suppose then, that λ_0 is an eigenvalue of operator A^*, G_0 is the corresponding eigenvector, and $G_1, G_2, \ldots, G_k$ is a Jordan chain of vectors. (There can be only finitely many vectors G_i because A^* is compact).

So, $A^*G_0=\lambda_0 G_0$, $A^*G_1=\lambda_0 G_1+G_0, \ldots, A^*G_k=\lambda_0 G_k+G_{k-1}$. It can be established directly that

$$(I-\bar{\lambda}A^*)^{-1}G_0=\frac{G_0}{1-\bar{\lambda}\lambda_0} \tag{7}$$

$$\begin{aligned}(I-\bar{\lambda}A^*)^{-1}G_1&=\frac{\bar{\lambda}}{1-\bar{\lambda}\lambda_0}(I-\bar{\lambda}A^*)^{-1}G_0+\frac{G_1}{1-\bar{\lambda}\lambda_0}\\ \vdots\qquad & \quad\vdots\\ (I-\bar{\lambda}A^*)^{-1}G_k&=\frac{\bar{\lambda}}{1-\bar{\lambda}\lambda_0}(I-\bar{\lambda}A^*)^{-1}G_{k-1}+\frac{G_k}{1-\bar{\lambda}\lambda_0}.\end{aligned} \tag{8}$$

Now (6) can be obtained directly from (7) and (8). So the proof of the Lemma is completed.

Notice that

$$\lim_{\substack{\lambda\to\infty\\ \lambda\in F_\varepsilon^c}}\|T(I-\lambda A)^{-1}\|=0 \tag{9}$$

because of the Lemma, where $T=S(I+S)^{-1}$ and the convergence is uniform on F_ε^c.

Proof of the Theorem. The proof of the Theorem is now based on an idea of Keldysh [4].

(a) From $B=A(I+S)$ it follows

$$(I-\lambda B)^{-1}=(I+S)^{-1}(I-T(I-\lambda A)^{-1})^{-1}(I-\lambda A)^{-1},$$
$$\text{where}\quad T=S(I+S)^{-1}. \tag{10}$$

Because of the Lemma, there exists $r>0$ so that for $|\lambda|>r$ and $\lambda\in F_\varepsilon^c$ one has

$$\|T(I-\lambda A)^{-1}\|\leqslant a<1\qquad (a=\text{const}).$$

So the formula (10) is correct and implies part (a) of the result and estimate

$$\|(I-\lambda B)^{-1}\|\leqslant\frac{\|(I+S)^{-1}\|}{(1-a)\sin\varepsilon} \tag{11}$$

for $\lambda\in F_\varepsilon^c$ and $|\lambda|>r$.

(b) Let $\mathscr{S}$ be the closed span of all generalized eigenvectors of the operator B.

Clearly $\mathscr{S} \subset \overline{\mathscr{R}(B)}$. Let P be an orthogonal projector of space $\mathscr{H}$ on $\mathscr{S}$ and $Q = I - P$. Thus $B_1 = QBQ$ is a Volterra operator [2], so the operator function $(I - \lambda B_1)^{-1}$ is an entire function and [2] one has

$$(I - \lambda B_1)^{-1} = Q(I - \lambda B)^{-1} Q + P. \tag{12}$$

Chose $\varepsilon > 0$ so that $\theta + \varepsilon < \pi/2p$. Since $A \in C_p$ one has $B \in C_p$ and $B_1 \in C_p$ and [2]

$$\ln \|(I - \lambda B_1)^{-1}\| = O(|\lambda|^p). \tag{13}$$

From (11), (12), and (13) it follows that the function $(I - \lambda B_1)^{-1}$ is bounded on boundary of sector F_ε and since $2(\theta + \varepsilon) < \pi/p$, by the theorem of Phragmen, Lindelof, and Liouville, follows

$$B_1 = 0 \qquad \text{or} \qquad QBQ = 0.$$

Then, we have $QB = 0$ or $B^* Q = 0$, i.e., $(I + S^*) A^* Q = 0$. Thus

$$A^* Q = 0. \tag{14}$$

From (14) follows $Q\mathscr{H} \subset \operatorname{Ker} A^*$, or $(Q\mathscr{H})^\perp \supset (\operatorname{Ker} A^*)^\perp$. Since $(Q\mathscr{H})^\perp = P\mathscr{H} = \mathscr{S}$ and $(\operatorname{Ker} A^*)^\perp = \overline{\mathscr{R}(A)}$ we have

$$\overline{\mathscr{R}(A)} \subset \mathscr{S}.$$

But, because of the assumption $-1 \in \rho(S)$, one has $\overline{\mathscr{R}(B)} = \overline{\mathscr{R}(B)}$ so that

$$\overline{\mathscr{R}(B)} \subset \mathscr{S}. \tag{15}$$

Since $\mathscr{S} \subset \overline{\mathscr{R}(B)}$, (15) implies $\mathscr{S} = \overline{\mathscr{R}(B)}$ i.e.,

$$\mathscr{S} = \overline{\mathscr{R}(A)}.$$

This completes the proof of the Theorem.

Remark 1. If $p \geqslant 1$ the assumption $\operatorname{Ker} A \subset \operatorname{Ker} S$ is not necessary, In that case, by Lidsky's theorem [2] $\operatorname{Ker} A = \{0\}$ and condition $\operatorname{Ker} A = \operatorname{Ker} S$ is satisfied.

Remark 2. If $p \geqslant 1$ the theorem can be formulated in the following way: If T is a compact operator and $A \in C_p$ is such that

$$\{(Af, f) : f \in \mathscr{H}\} \subset \Omega = \left\{\lambda : |\arg \lambda| \leqslant \theta < \frac{\pi}{2p}\right\}$$

then the system of eigenvectors and associated vectors [2] of the linear pencil $L(\lambda) = I - T - \lambda A$ is complete in $\mathscr{H}$.

EXAMPLE. The operator L generated by the differential expression

$$l(y) = -y'' + p_1(x)\, y' + p_2(x)\, y$$

and boundary condition is m-sectorial if for example

$$y(0) = 0, \qquad y(\pi) = 0$$

$$\max_{0 \leqslant x \leqslant \pi} |p_1(x)|^2 \leqslant \min_{0 \leqslant x \leqslant \pi} |p_2(x)|,$$

where p_1, p_2 are real continuous functions on $[0, \pi]$. Suppose that the equation $Ly = 0$ has only the trivial solution $y = 0$.

If $K(\cdot, \cdot)$ is a continuous function on $[0, \pi] \times [0, \pi]$ so that

$$\int_0^\pi \int_0^\pi |(K(x, s)|^2 \, dx\, ds < 1$$

and the boundary value problem

$$-y'' + p_1(x)\, y' + p_2(x)\, y = \lambda y + \lambda \int_0^\pi K(x, s)\, y(s)\, ds$$

$$y(0) = y(\pi) = 0$$

has simple eigenvalues, then the corresponding system of eigenvectors is complete in $L^2(0, \pi)$.

The proof follows directly by applying the theorem to the operator $A = L^{-1}(I + S)$ where $S: L^2(0, \pi) \to L^2(0, \pi)$ is the linear operator defined by

$$Sf(x) = \int_0^\pi K(x, s)\, f(s)\, ds.$$

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